Lecture 1:

Scaler first order differential equations:

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R} \times \mathbb{R}$ then SCALER DIFFERENTIAL EQUATION:

$$\dot{x}(t) = f(t, x(t)) \tag{A.1}$$

with t independent and x dependent variable.

$$\text{meaning of the dot: } \dot{x}(t) = \frac{dx}{dt}(t) \tag{A.2}$$

$$\textsf{Notation: } \dot{x}(t) = f(t, x) \tag{A.3}$$

Definition A.1:
A SOLUTION TO A DIFFERENTIAL FUNCTION $x : I \to \mathbb{R}$ where $I \subset \mathbb{R}$ if:
1: x is differentiable.
2: $(t, x(t)) \in D$ for all $t \in I$
3: $\dot{x}(t) = f(t, x(t))$ for all $t \in I$

Problem A.1:
INITIAL VALUE PROBLEM:
Given $f : D \to \mathbb{R}$ for some $D \subset \mathbb{R} \times \mathbb{R}$ and point (t_0, x_0) find such a solution $x : J \to \mathbb{R}$ s.t.

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \tag{A.6}$$

New form (part 1):

1:

$$\dot{x}(t) = f(t) \tag{A.9}$$

Then the UNIQUE SOLUTION: for a initial condition $x(t_0) = x_0$ given by:

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau) d\tau \tag{A.10}$$

2: AUTONOMOUS EQUATIONS:

$$\dot{x}(t) = g(x(t)) \tag{A.11}$$

where g continuous on interval X, so $D = \mathbb{R} \times X$.
We can show that these equations are TIME-INVARIANT by using that

$$H(x) = t + \tau \Leftrightarrow \frac{dH}{dx}(x) = \frac{1}{g(x)} \Leftrightarrow \frac{d}{dt}(h(x)) = \frac{dH}{dx}(x)\dot{x} = \frac{1}{g(x)}\dot{x} = 1 \tag{A.12,A.13,A.14}$$

Lecture 2:

New form (part 2):

1: SEPERABLE EQUATIONS:

$$\dot{x}(t) = f(t)g(x(t)) \tag{A.19}$$

CLASS OF SOLUTIONS given by:

$$H(x) = F(t) + c \tag{A.21}$$

with $c \in \mathbb{R}$ and $\frac{dF}{dt}(t) = f(t)$
REASONING:

$$\frac{d}{dt}(H(x)) = \frac{1}{g(x)}\dot{x} = f(t) \tag{A.22}$$

Therefore:

$$\frac{dx}{dt} = f(t)g(x) \Leftrightarrow \frac{1}{g(x)}dx = f(t)dt \Leftrightarrow \int \frac{1}{g(x)}dx = \int f(t)dt \Leftrightarrow H(x) = F(t) + c \quad \text{(A.23,A.24,A.25)}$$

Linear scaler differential equations,linearity:

$a, b : I \to \mathbb{R}$ then LINEAR DIFFERENTIAL EQUATION

$$\dot{x}(t) = a(t)x(t) + b(t) \tag{A.37}$$

HOMOGENEOUS: $b(t) = 0$ for all t v.s. NONHOMOGENEOUS: $b(t) \neq 0$

DIFFERENTIAL OPERATOR:

$$L(x) = \dot{x} - a(t)x \tag{1}$$

This operator is indeed linear: x', x two functions and $\alpha', \alpha \in \mathbb{R}$ then we see that:

$$\begin{aligned} L(\alpha x + \alpha' x') &= \frac{d}{dt}\{\alpha x + \alpha' x'\} - a(t)(\alpha x + \alpha' x') \\ &= \alpha \dot{x} + \alpha' \dot{x}' - a(t)\alpha x - a(t)\alpha' x' = \alpha L(x) + \alpha' L(x') \end{aligned} \tag{A.39}$$

When we rewrite this we obtain that:

$$L(x) = b(t) \tag{A.40}$$

Homogeneous equation:

$$\dot{x}(t) = a(t)x(t) \tag{A.41}$$

which we can rewrite to:

$$H(x) = \int \frac{1}{x}dx = \ln|x|, \qquad F(t) = \int a(t)dt \tag{A.42}$$

So any solution where $C = \pm e^c$ or $C = 0$ given by:

$$x(t) = Ce^{F(t)} \tag{A.44}$$

Lemma A.1.
Consider initial value problem:

$$\dot{x}(t) = a(t)x(t), \qquad x(t_0) = x_0 \tag{A.48}$$

Where $a : I \to \mathbb{R}$ and $t_0 \in I$ then the unique solution:

$$x(t; t_0, x_0) = x_0 e^{F(t)} \qquad F(t) = \int_{t_0}^{t} a(\tau)d\tau \tag{A.49}$$

PROOF:
Introduce $z(t) = x(t) - x'(t)$ where $x'(t)$ second solution. So then

$$\int_{t_0}^{t} \frac{d}{d\tau}\{e^{-F(\tau)}z(\tau)\}d\tau = e^{-F(t)}z(t) - e^{-F(t_0)}z(t_0) = 0 \tag{A.51}$$

So indeed solution and indeed unique. $a : I \to \mathbb{R}$ and $t_0 \in \mathbb{R}$ then UNIQUE SOLUTION: for $t \in I$

$$x(t; t_0, x_0) = x_0 e^{F(t)}, \qquad F(t) = \int_{t_0}^{t} a(\tau)d\tau \tag{A.49}$$

Non-homogeneous equations:

$$\dot{x}(t) = a(t)x(t) + b(t) \tag{A.59}$$

We find the class of solution by VARIATION OF CONSTANTS:

$$\begin{aligned} z(t) &= e^{-F(t)}x(t) \Rightarrow \dot{z}(t) = e^{-F(t)}b(t) \\ z(t) &= \int e^{-F(t)}b(t)dt + C \end{aligned} \quad \text{(A.60,A.61,A.62)}$$

So then we get that:

$$x(t) = Ce^{F(t)} + e^{F(t)} \int e^{-F(t)}b(t)dt \tag{A.63}$$

Lemma A.2.
Consider initial value problem where $a, b : I \to \mathbb{R}$ and $t_0 \in I$ s.t.

$$\dot{x}(t) = a(t)x(t) + b(t) \qquad x(t_0) = x_0 \tag{A.64}$$

Then UNIQUE SOLUTION for $t \in I$ and F as in (A.49)

$$x(t; t_0, x_0) = x_0 e^{F(t)} + e^{F(t)} \int_{t_0}^{t} e^{-F(\tau)}b(\tau)d\tau \tag{A.65}$$

Homogeneous and particular solutions.

We can write (A.59) as

$$x(t) = x_h(t) + x_p(t) \tag{A.67}$$

where HOMOGENEOUS SOLUTION:

$$\dot{x}_h(t) = a(t)x_h(t) \tag{A.68}$$

annd PARTICULAR SOLUTION

$$\begin{aligned} \dot{x_p}(t) &= a(t)x_p(t) + b(t) \\ x_p(t) &= C_p e^{F(t)} + e^{F(t)} \int e^{-F(t)}b(t)dt \end{aligned} \quad \text{(A.69,A.70)}$$

Let $x_h(t) = C_h e^{F(t)}$ then we can write (A.67) as

$$x(t) = (C_h + C_p)e^{F(t)} + e^{F(t)} \int e^{-F(t)}b(t)dt \tag{A.71}$$

Example:

$\dot{x}(t) = -\frac{4}{t}x(t) + \frac{sin(t)}{t^4}$ where $t > 0$

1) HOMOGENEOUS $\frac{dx}{dt} = -\frac{4}{tx} \Rightarrow \int \frac{1}{x}dx = -4\int \frac{1}{t}dt \Rightarrow \ln|x| = -4\ln|t| + C \Rightarrow x(t) = \frac{c}{4}$

2) PARTICULAR: $z(t)t^4x(t) \Rightarrow \dot{z}(t) = \sin(t) \Rightarrow z(t) = -\cos(t)$

3) FINAL SOLUTION: $x(t) = \frac{C - \cos(t)}{t^4}$

(A.71 till A.76)

Lecture 3:

SYSTEM OF DIFFERENTIAL EQUATIONS:

$$\begin{aligned}\dot{x}_1(t) &= f_1(t, x_1(t), \ldots, x_n(t)) \\ &\vdots \\ \dot{x}_n(t) &= f_n(t, x_1(t), \ldots, x_n(t))\end{aligned} \tag{A.79}$$

When we define

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad f(t,x) = \begin{bmatrix} f_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{bmatrix} \qquad \text{(A.80,A.81)}$$
$$\dot{x}(t) = f(t, x(t))$$

Definition A.2.
$I \subset \mathbb{R}$. $x : I \to \mathbb{R}^n$ SOLUTION to (A.80,A.81) if x differentiable, $(t, x(t)) \in D$ for all $t \in I$ holds for all $t \in I$
Problem A.2:
$f : D \to \mathbb{R}^n$ for $D \subset \mathbb{R} \times \mathbb{R}^n$ and a point $(t_0, x_0) \in D$ find solution $x : I \to \mathbb{R}^n$ s.t.

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0 \tag{A.82}$$

SOLUTION INITIAL VALUE PROBLEM:

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \tag{A.83}$$

Recall: $|x| = \sqrt{x^T x} = \sqrt{x_1^2 + \ldots + x_n^2}$

Definition A.3.
$f : D \to \mathbb{R}^n$ with $D \subset \mathbb{R} \times \mathbb{R}^n$ is LIPSCHITZ in x at a point $(t'x') \in D$ if $\exists L > 0 \& r > 0$ s.t.

$$|f(t,x) - f(t,x'| \leq L|x - x'| \tag{A.85}$$

$\forall (t,x) : |x - x'| < r, |t - t'| < r, (t,x) \in D$. f Lipschtiz for all $(t', x') \in D$, it is said to be locally Lipschitz on D.

Theorem A.3.
$f : D \to \mathbb{R}^n$ with $D \subset \mathbb{R} \times \mathbb{R}^n$ continuous, locally Lischitz on D¿ If $(t_0, x_0) \in D$ then exists unique solution (A.82), can be extended left and right up to boundaries of D
PROOF:,Remark A.4:
By METHOD OF SUCCESSIVE APPROXIMATIONS:

$$x^{(k+1)}(t) = x_0 + \int_{t_0}^{t} f(\tau, x^k(\tau))d\tau \tag{A.86}$$

Remark A.5:
For $f : \mathbb{R} \to \mathbb{R}$ (A.85) will be:

$$\frac{|f(x) - f(x')|}{|x - x'|} = \left| \frac{f(x) - f(x')}{x - x'} \right| \leq L \tag{A.87}$$

Remark A.6:

$$y^{(k)}(t) = \frac{d^k y}{dt^k}(t) \tag{A.88}$$

So then n th order diferential equation:

$$y^{(n)}(t) = f(t, y(t), \dot{y}(t), \ldots, y^{(n-1)}(t)) \tag{A.89}$$

When we say that $x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T = \begin{bmatrix} y & \ldots & y^{(n-1)} \end{bmatrix}^T$ we see that we get:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t, x_1, \ldots, x_{n-1}) \end{bmatrix} \tag{A.91}$$

Systems:

LINEAR SYSTEMS WITH INPUTS AND OUTPUTS:

$$\Sigma = \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{STATE EQUATION} \\ y(t) = Cx(t) + Du(t) & \text{OUTPUT EQUATION} \end{cases} \tag{1.1}$$

t indep. var. time.STATE: $x(t) \in \mathbb{R}^n$.INPUT: $u(t) \in \mathbb{R}^m$.OUTPUT: $y(t) \in \mathbb{R}^p$.
SYSTEM MATRIX: $A \in \mathbb{R}^{n \times n}$ and INPUT MATRIX: $B \in \mathbb{R}^{n \times m}$.
OUTPUT MATRIX: $C \in \mathbb{R}^{p \times n}$ and FEEDTHROUGH MATRIX: $D \in \mathbb{R}^{p \times m}$

NONLINEAR SYSTEM:

$$\Sigma_{\text{nl}} = \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases} \tag{1.14}$$

Definition 1.1:
Consider Σ_{nl} for constant $u(t) = \overline{u}$,Then $\overline{x} \in \mathbb{R}^n$ EQUILIBRIUM for $\overline{u}$ if

$$f(\overline{x}, \overline{u}) = 0 \tag{2}$$

Means that $x(t) = \overline{x}$ constant solution. Corresponding output $\overline{y} \in \mathbb{R}^p$ as $\overline{y} = h(\overline{x}, \overline{u})$

DEVIATION FROM THE EQUILIBRIUM POINT:

$$\tilde{x}(t) = x(t) - \overline{x} \quad \tilde{u}(t) = u(t) - \overline{u} \quad \tilde{y}(t) = y(t) - \overline{y} \tag{1.22,1.23}$$

DYNAMICS:

$$\dot{\tilde{x}} = \dot{x}(t) - 0 = f(x(t), u(t)) = f(\overline{x} + \tilde{x}(t), \overline{u} + \tilde{u}(t)) \tag{1.24}$$

For $\tilde{x}, \tilde{u}$ small:

$$f(\overline{x} + \tilde{x}, \overline{u} + \tilde{u}) \approx f(\overline{x}, \overline{u}) + \frac{\partial f}{\partial x}(\overline{x}, \overline{u})\tilde{x} + \frac{\partial f}{\partial u}(\overline{x}, \overline{u})\tilde{u} \tag{1.25}$$

When we use that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{1.26}$$

Similair for $\frac{\partial f}{\partial u}, \frac{\partial h}{\partial u}, \frac{\partial h}{\partial x}$ so then we can obtain:
Definition 1.2:
Let $(\overline{x}, \overline{u})$ equilibrium of (1.14) then:

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\tilde{u}(t) \\ \tilde{y}(t) &= C\tilde{x}(t) + D\tilde{u}(t) \end{aligned} \tag{1.30}$$

with $\tilde{x}(t) \in \mathbb{R}^n$ and $\tilde{u}(t) \in \mathbb{R}^m$ and $\tilde{y}(t) \in \mathbb{R}^p$ and

$$A = \frac{\partial f}{\partial x}(\overline{x}, \overline{u}), \quad B = \frac{\partial f}{\partial u}(\overline{x}, \overline{u}), \quad C = \frac{\partial h}{\partial x}(\overline{x}, \overline{u}), \quad D = \frac{\partial h}{\partial u}(\overline{x}, \overline{u}) \tag{1.31}$$

is called LINEARIZATION of (1.14) around equilibrium $(\overline{x}, \overline{u})$

Lecture 4:

Homogeneous linear systems:

Consider

$$\dot{x}(t) = Ax(t) \tag{2.1}$$

with the initial value problem stated in problem A.2, so (A.82).
We see that the unique solution is:

$$\begin{aligned} x(t;t_0,x_0) &= x_0 e^{a(t-t_0)} = x_0 + \int_{t_0}^{t} Ax(\tau)d\tau \\ x^{(k+1)}(t) &= x_0 + \int_{t_0}^{t} Ax^{(k)}(\tau)d\tau \end{aligned} \tag{2.3,2.4,2.5}$$

Idea: approximate $x^{(k)}(t)$

$$\begin{aligned} x^{(1)}(t) &= x_0 + \int_{t_0}^{t} Ax_0 d\tau = (I + A(t-t_0))x_0 \\ x^{(2)}(t) &= x_0 + \int_{t_0}^{t} (I + A(t-\tau_0))x_0 d\tau = (I + A(t-t_0) + \frac{1}{2}A^2(t-t_0)^2)x_0 \\ &\vdots \\ x^{(k)}(t) &= \left(\sum_{l=0}^{k} \frac{A^l(t-t_0)^l}{l!}\right)x_0 \end{aligned} \tag{2.6,2.7}$$

Now we first define $\|\circ\|_2$ we will do this by:

$$\|A\| = \sup\left\{\frac{|A|}{|x|} | x \neq 0\right\} = \sup\{|Ax| | |x| = 1\} \tag{2.9}$$

Where $|x| = \sqrt{x^*x}$ where x^* the Hermitian transpose of x

Lemma 2.1:
$A \in \mathbb{C}^{n\times n}$ and $(A)_{ij}$ be any element of A. Then

$$|(A)_{ij}| \leq \|A\| \tag{2.10}$$

PROOF:
e_i is i th column if I_n so then:

$$Ae_j = \begin{bmatrix} (A)_{1j} \\ (A)_{2j} \\ \vdots \\ (A)_{nj} \end{bmatrix} \Rightarrow |(A)_{ij}| \leq |Ae_j| \leq \|A\| \tag{2.11,2.12}$$

Lemma 2.2:
$A, B \in \mathbb{C}^{n\times n}$ then

$$\|AB\| \leq \|A\|\|B\| \tag{2.13}$$

PROOF:
Definition 2.9: $|Ax| \leq \|A\||x|$ for any $x \in \mathbb{C}$ so therefore

$$|ABx| \leq \|A\||Bx| \leq \|A\|\|B\||x| \quad \forall x \in \mathbb{C}^n \tag{2.14}$$

Lemma 2.3:
The matrix series $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ converges for every $At \in \mathbb{C}^{n\times n}$
PROOF:

$$\begin{aligned} \left|\frac{t^k}{k!}(A^k)_{ij}\right| \leq \left\|\frac{t^k}{k!}A^k\right\| = \frac{\left\|(At)^k\right\|}{k!} \leq \frac{\|At\|^k}{k!} \\ \sum_{k=0}^{\infty}\left|\frac{t^k}{k!}(A^k)_{ij}\right| \leq \sum_{k=0}^{\infty}\left\|\frac{t^k}{k!}A^k\right\| = e^{\|At\|} < \infty \end{aligned} \tag{2.15,2.16,2.17}$$

Definition 2.1: MATRIX EXPONENTIAL of At with $A \in \mathbb{C}^{n\times n}$:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \tag{2.18}$$

Lemma 2.4: $t \mapsto e^{At}$ for $A \in \mathbb{R}^{n\times n}$ then $\frac{d}{dt}\{e^{At}\} = Ae^{At}$
PROOF:
$\frac{d}{dt}\left\{e^{At}\right\} = \frac{d}{dt}\left\{I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}\right\} = \sum_{k=1}^{\infty} \frac{d}{dt}\left\{\frac{A^k t^k}{k!}\right\} = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A\sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!} = Ae^{At} = e^{At}A$
Theorem 2.5:
Consider:
$\dot{x}(t) = Ax(t)$ and $x(t_0) = x_0$ for $A \in \mathbb{R}^{n\times n}$ then the unique solution for $t \in \mathbb{R}$ is given by:

$$x(t; t_0, x_0) = e^{A(t-t_0)}x_0 \tag{2.21}$$

Nonhomogeneous linear systems:

$\dot{x}(t) = Ax(t) + Bu(t)$
Remark 2.2: continuous input function.
INITIAL VALUE PROBLEM: for given input function and initial condition

$$\dot{x}(t) = Ax(t) + Bu(t) \quad x(t_0) = x_0,\ u : I \to \mathbb{R}^m \tag{2.66}$$

Theorem 2.11:
for $t \in I$ the unique solution to (2.66) is given by:

$$x(t; t_0, x_0, u) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{2.67}$$

PROOF:
idea derrive solution using variation of constants:
$z(t) = e^{-At}x(t) \Leftrightarrow e^{At}z(t) = x(t)$
$\dot{z}(t) = \frac{d}{dt}\{e^{-At}x(t)\} = -Ae^{-At}x(t) + e^{-At}\dot{x}(t) = -Ae^{-At}x(t) + e^{-At}(Ax(t) + Bu(t))$
$= -Ae^{-At}x(t) + Ae^{-A(t)}x(t) + e^{-At}Bu(t) = e^{-At}Bu(t)$
$z(t) = z(t_0) + \int\limits_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau = e^{-At_0}x_0 + \int\limits_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$
$x(t) = e^{At}e^{-At_0}x_0 + \int\limits_{t_0} e^{At}e^{-A\tau}Bu(\tau)d\tau$
$= e^{A(t-t_0)}x_0 + \int\limits_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$ which is also unique.
And because this one is unique, the solution of the homogeneous system is unique.

Remark 2.3:
Consider output equation

$$y(t) = Cx(t) + Du(t) \tag{2.73}$$

then for all $t \in I$ and for initial condition (t_0, x_0) we have

$$y(t; t_0, x_0, u) = Ce^{A(t-t_))}x_0 + \int\limits_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{3}$$

Theorem 2.12:
Consider $\dot{x}(t) = Ax(t) + Bu(t)$. THen for (t_0, x_0) and (t_0, x_0') with $x_0, x_0' \in \mathbb{R}^n$ two initial conditions and $u, u' : I \to \mathbb{R}^m$ two input function with $t_0 \in I$ then
$x(t; t_0, \alpha x_0, \alpha' x_0', \alpha u + \alpha' u') = \alpha x(t; t_0, x_0, u) + \alpha' x(t; t_0, x_0', u')$ for all $\alpha, \alpha' \in \mathbb{R}$

Theorem 2.13:TIME-INVARIANCE: for any (t_0, x_0) and $u : \mathbb{R} \to \mathbb{R}^m$ then:
$x(t; t_0, x_0, u) = x(t - t_0; 0, x_0, u_{t_0})$ with $u_{t_0}(t) = u(t + t_0)$ and $t \in \mathbb{R}$

Lecture 5:

Computation of the matrix exponential:

For $A \in \mathbb{R}^{n\times n}$ ($A \in \mathbb{C}^{n\times n}$)
$e^{At} = \sum\limits_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{1}{2}A^2t^2 + \dots$

Lemma 2.6.
Let $T \in \mathbb{C}^{n\times n}$ be nonsingular and $A \in \mathbb{C}^{n\times n}$ then:

$$e^{TAT^{-1}t} = Te^{At}T^{-1} \quad \forall t \tag{2.37}$$

Remark 2.1:
$A \in \mathbb{R}^{n\times n}$ diagonalizable $\Rightarrow$ exists n lin. independent eigenvectors $v_i \in \mathbb{C}^m$ s.t.

$$\begin{gathered} AT = T\Lambda \\ T = \begin{bmatrix} v_1 & \dots & v_i \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \end{gathered} \tag{2.42,2.43}$$

So then

$$e^{A(t-t_0)} = e^{T\Lambda T^{-1}(t-t_0)} = Te^{\Lambda(t-t_0)}T^{-1} \Rightarrow x(t;t_0,x_0) = e^{A(t-t_0)}x_0 = \sum_{i=1}^{n} c_i v_i e^{\lambda i(t-t_0)} \tag{2.44,2.45}$$

Lemma 2.7.
$a, b \in \mathbb{C}^{n\times n}$ s.t. $AB = BA$ then for all $t \in \mathbb{R}$

$$e^{At}B = Be^{At} \tag{2.46}$$

Lemma 2.8.
For any $A \in \mathbb{C}^{n\times n}$

$$\begin{aligned} &\left(e^{At}\right)^{-1} = e^{-At} \\ &\text{if } AB = BA \Rightarrow e^{At}e^{Bt} = e^{(A+B)t} \\ &\forall t, s \in \mathbb{R}\; e^{At}e^{As} = e^{A(t+s)} \end{aligned} \tag{2.47,2.48,2.49}$$

The Jordan canonical form:

Definition 2.2.
Jordan block matrix $J_k(\lambda) \in \mathbb{C}^{k\times k}$ matrix given by:

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \tag{2.62}$$

Note: $J_k(\lambda) = \lambda I + N$ with $N^k = 0$

Lemma 2.9.
Consider $J_k(\lambda)$ for positive integer k and $\lambda \in \mathbb{C}$ then:

$$e^{J_k(\lambda)t} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k-2}}{(k-2)!} & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & & & \\ & \ddots & \ddots & & & \\ & & & & t & \frac{t^2}{2!} \\ & & & & 1 & t \\ & & & & & 1 \end{bmatrix} \tag{2.62}$$

Theorem 2.10:
For any $A \in \mathbb{R}^{n\times n}$ there exists a nonsingular matrix,$T \in \mathbb{C}^{n\times n}$ s.t. $A = TJT^{-1}$ with

$$J = \begin{pmatrix} J_{k_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{k_r}(\lambda_r) \end{pmatrix} \tag{2.63}$$

Where $\lambda_i \in \sigma(A)$ and $i = 1, \dots, r$ and $n = k_1 = \dots + k_r$. Conversely, let $\lambda \in \sigma(A)$. Then $\lambda = \lambda_i$ for some $i \in \{1, \dots, r\}$

Computation of the matrix exponential:

1: Compute Jordan canonical form: $A = TJT^{-1}$
2: for each Jordan block $(J_{k_i}(\lambda(i))$ compute $e^{J_{k_i}(\lambda_i)t}$
3: Compute e^{At} using:

$$\begin{aligned} e^{At} &= e^{TJT^{-1}t} \\ &= Te^{Jt}T^{-1} = T \begin{bmatrix} e^{J_{k_1}(\lambda_1)t} & & \\ & \ddots & \\ & & e^{J_{k_r}(\lambda_r)t} \end{bmatrix} T^{-1} \end{aligned} \tag{2.64}$$

Lecture 6:

Consider

$$\dot{x}(t) = Ax(t) \tag{3.1}$$

Definition 3.1.
The system (3.1) is called

1. STABLE if every solution bounded for $t \geq 0$. So for any $x_0 \in \mathbb{R}$ exists $M > 0$ s.t.

$$|x(t; x_0)| = \left|e^{At}x_0\right| \leq M \quad t \geq 0 \tag{3.2}$$

2. ASYMPTOTICALLY STABLE: every solution tends to zero for $t \to \infty$ so for $x_0 \in \mathbb{R}$

$$\lim_{t\to\infty}(t; x_0) = \lim_{t\to\infty} e^{At}x_0 = 0 \tag{3.3}$$

Remark 3.1: holds also for (1.1)

Consider $t \mapsto t^k e^{\lambda t}$ for nonnegative integer k and $\lambda \in \mathbb{C}$
Lemma 3.1:
If $\mathrm{Re}(\lambda) < 0$ then

$$\lim_{t\to\infty} t^k e^{\lambda t} = 0 \tag{3.4}$$

And $\left|t^k e^{\lambda t}\right|$ bounded $\forall t \geq 0$
Lemma 3.2:
for any $\alpha \in \mathbb{R}$ s.t. $\mathrm{Re}(\lambda) < \alpha$,

$$\left|t^k e^{\lambda t}\right| \leq M e^{\alpha t} \quad \forall t \geq 0 \tag{3.5}$$

PROOF:
Define $p(t) = e^{-\alpha t} t^k e^{\lambda t} = t^l e^{(\lambda-\alpha)t}$
Then $\mathrm{Re}(\lambda - x) < 0$ and by this: $|p(t)| \leq M \Rightarrow e^{\alpha t}|p(t)| = |e^{\alpha t}p(t)| = \left|t^k e^{\lambda t}\right| \leq Me^t$

Denote:

$\mathbb{C} = \{z \in \mathbb{C} | \mathrm{Re}(z) < 0\}$ and $\overline{\mathbb{C}} = \{z \in \mathbb{C} | \mathrm{Re}(z) \leq 0\}$

Theorem 3.3:
Consider (3.1) then:
1) (3.1) is stable iff

$$\sigma(A) \subset \overline{\mathbb{C}}_- \tag{3.7}$$

and every λ with $\mathrm{Re}(\lambda) = 0$ is semisimple: equal algebraic and geometric multiplicity
2: (3.1) asymptotically stable iff

$$\sigma(A) \subset \mathbb{C}_- \tag{3.8}$$

Exists $M, \gamma > 0$ s.t.

$$\left\|e^{At}\right\| \leq Me^{-\gamma t} \quad \forall t \geq 0 \tag{3.9}$$

PROOF:
By contraposition: let $\lambda \in \sigma(A)$ with $\mathrm{Re}(\lambda) \geq 0$.
Denote $Av = \lambda v$ then $x(t) = e^{\lambda t}v$ solution of $\dot{x}(t) = Ax(t)$ Namely:
$\dot{x}(t) = \lambda e^{\lambda t}v = \lambda v e^{\lambda t} = Ave^{\lambda t} = Ax(t)$
In fact, $x(t; v) = e^{\lambda t}v$ however $\lim_{t\to\infty} x(t; v) \neq 0$ so $\dot{x}(t) = Ax(t)$ is not asymptotically stable.

Routh-Hurwitz criterion:

Definition 3.2:
A polynomial

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{3.17}$$

with $a_i \in \mathbb{R}$ and $a_n \neq 0$ called stable if all its roots $(p(\lambda) = 0)$ have negative real parts: $\sigma(p) \subset \mathbb{C}$

Theorem 3.4:
ROUTH-HURWITZ THEOREM:
p as (3.17) with $a_0, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Then p stable iff:

$$\begin{aligned} &\mathbf{1)}\, a_{n-1} \neq 0 \\ &\mathbf{2)}\, a_n a_{n-1} > 0 \\ &\mathbf{3)}\, q(s) = a_{n-1}p(s) - a_n(a_{n-1}s^n + a_{n-3}s^{n-2} + \ldots) \quad \text{is stable} \end{aligned} \tag{3.19}$$

Lemma 3.5
Let p be

$$p(s) = a_2 s^2 + a_1 s + a_0 \tag{3.29}$$

with $a_0, a_1, a_2 \in \mathbb{R}$ and $a_n \neq 0$. Then p stable iff al coefficients nonzero and same sign.

Lemma 3.6:
let p be as (3.17) with $a_0, \ldots, a_n \in \mathbb{R}$, p stable, then all coefficients nonzero and same sign.

Interval polynomials:

Define

$$\mathcal{P}(s) = \{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 | a_i^- \leq a_i \leq a_i^+ \forall i \in \{0, 1 \ldots, n\}\} \tag{3.33}$$

Definition 3.3:
the set $\mathcal{P}$ stable if p stable for all $p \in P$.

Theorem 3.7:
Set $\mathcal{P}$ stable iff the following four polynomials stable:

$$\begin{aligned} p^{++}(s) &= a_0^+ + a_1^+ s + a_2^+ s^2 + a_3^+ s^3 + \ldots \\ p^{+-} &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + \ldots \\ p^{-+} &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + \ldots \\ p^{--} &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + \ldots \end{aligned} \tag{3.34,3.35,3.36,3.37}$$

Lecture 7:

$$\Sigma(A,B) : \dot{x}(t) = Ax(t) + Bu(t) \tag{4.1}$$

Recall:
Unique solution given by:

$$x(t; x_0, u) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{4.2}$$

Definition 4.1.
(4.1) is CONTROLLABLE if for any $x(0) = x_0 \in \mathbb{R}^n$ and any final state $x_f \in \mathbb{R}^n$ there $\exists u : [0,T] \to \mathbb{R}^m$ s.t. $x_f = x(T; x_0, u)$

Definition 4.2.
The state $x_f \in \mathbb{R}^n$ is REACHABLE at time $T > 0$ if there exists $u : [0,T] \to \mathbb{R}^m$ s.t. $x(T; 0, u) = x_f$
REACHABLE SUBSPACE $\mathcal{W}_T$ at some time $T > 0$ s.t.:

$$\begin{aligned}\mathcal{W}_T &= \{x_f \in \mathbb{R}^n | x_f \text{ is reachable at } T\} \\ &= \left\{ \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau | u : [0,T] \to \mathbb{R}^m \right\}\end{aligned} \tag{4.7}$$

Definition 4.3.
(4.1) REACHABLE if for any $x_f \in \mathbb{R}^n$ is reachable: $\mathcal{W}_T = \mathbb{R}^n$
Lemma 4.1.
(4.1) controllable iff (4.1) reachable.

Theorem 4.2:
$v \in \mathbb{R}^n$ and $T > 0$ then:
1) $v^T x = 0$ for all $x \in W_T$
2) $v^T e^{At} B = 0$ for all $0 \le t \le T$
3) $v^T A^k B = 0$ for $k = 1, \ldots$
4) $v^T \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = 0$
Where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ where $m < n$

PROOF $1 \Rightarrow 2$
Let $v^T x = 0$ for all $x \in w_T$ then:
$0 = \int_0^T v^T e^{A(t-\tau)}Bu(\tau)d\tau$ for every $u : [0,T] \to \mathbb{R}^m$
Choose $u(t) = B^T e^{A^T(T-\tau)}v$
$0 = \int_0^T v^t eA(T-\tau)BB^T e^{A^T(T-\tau)}v d\tau = \int_0^T \left\| B^T e^{A^T(T-\tau)}v \right\|^2 d\tau \Leftrightarrow B^T e^{A^T t}v = 0$ for $0 \le t \le T$
So $v^T e^{At}B = 0$ for $0 \le t \le T$

Corollary 4.3:
$\mathcal{W}_T$ independent of T for $T > 0$. I tsatisfies:

$$\mathcal{W}_T = \operatorname{im}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \tag{4.12}$$

Image:

$$\operatorname{im} M = \{y \in \mathbb{R}^p | y = Mx \,\text{for some}\, x \in \mathbb{R}^q\} \tag{4.12}$$

Definition 4.4.
$A \in \mathbb{R}^{n\times n}$ and $\mathcal{V} \subset \mathbb{R}^n$. Then $\mathcal{V}$ is called $A-$,invariant if:

$$x \in \mathcal{V} \Rightarrow Ax \in \mathcal{V} \tag{4.14}$$

Notation: $A\mathcal{V} \subset \mathcal{V}$

Theorem 4.4.
reachable subspace $\mathcal{W}$ smallest $A-$ invariant subspace containing $\operatorname{im} B$

Corollary 4.5.
1) $\exists T > 0$ s.t Σ controllable at T
2) Σ controllable at T for all $T > 0$
3) rank $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$
4) $\mathcal{W} = \mathbb{R}^n$

Observability:

Consider

$$\Sigma(A, C) = \begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases} \tag{4.25}$$

Definition 4.5.
Consider (4.25). Two states $x_0 \& x_0' \in \mathbb{R}$. They are INDISTINGUISHABLE: on $[0, T]$ with $T > 0$ if

$$y(t; x_0, 0) = y(t; x_0', 0) \tag{4.30}$$

Definition 4.6.
(4.25) observable on $[0, T]$ if two states $x_0, x_0' \in \mathbb{R}^n$ are indistinguishable on $[0, T]$ only if $x_0 = x_0'$

Definition 4.7.
UNOBSERVABLE SUBSPACE: at time T denoted by $\mathcal{N}_T$

$$\mathcal{N}_T = \{x \in \mathbb{R}^n | x \,\text{is indistinguishable from 0 on}\, [0, T]\} \tag{4.31}$$

Theorem 4.6.
$\mathcal{N}_T$ independent of T for $T > 0$ so

$$\mathcal{N}_T = \ker \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{4}$$

Theorem 4.7.
unobservable subspace $\mathcal{N}$ is largest $A-$ invariant subspace contained in $\ker(C)$

Theorem 4.8.
Following equivalent:
1) $\exists T > 0$ s.t. $\Sigma(A, C)$ is obversable at T
2) $\Sigma(A, C)$ observable for all $T > 0$
3) rank $\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$
4 $\mathcal{N} = \{0\}$

Theorem 4.9.
System $\Sigma(A, B)$ in (4.1) controllable iff $\Sigma(A^T, B^T)$ in (4.25) is observable.

Lecture 8:

Canonical forms

Let $T \in \mathbb{R}^{n\times n}$ be nonsingular s.t.

$$\begin{aligned} &\overline{x}(t) = Tx(t) \\ &(4.1) \Rightarrow \Sigma(TAT^{-1}, TB) : \dot{\overline{x}}(t) = TAT^{-1}\overline{x}(t) + TBu(t) \\ &(4.25) \Rightarrow \Sigma(TAT^{-1}, CT^{-1}) := \begin{cases} \dot{\overline{x}}(t) = TAT^{-1}\overline{x}(t) \\ y(t) = CT^{-1}\overline{x}(t) \end{cases} \end{aligned} \tag{4.41,4.42,4.43}$$

Definition 4.8.
$\Sigma(A, B)$ and $\Sigma(\overline{A}, \overline{B})$ form: (4.1)SIMILAR if

$$\overline{A} = TAT^{-1}\overline{B} = TB \tag{4.44}$$

$\Sigma(A, C)$ and $\Sigma(\overline{A}, \overline{C})$ (4.25)SIMILAR if

$$\overline{A} = TAT^{-1}\overline{C} = CT^{-1} \tag{4.44}$$

Theorem 4.10:
$T \in \mathbb{R}^{n\times n}$ nonsingular then:
1) $\Sigma(A, B)$ controllable $\Leftrightarrow \Sigma(TAT^{-1}, TB)$ controllable.
2) $\Sigma(A, C)$ observable $\Leftrightarrow \Sigma(TAT^{-1}, CT^{-1})$ observable.

Theorem 4.11: Let $\Sigma(A, B)$ be unocontrollable, define $k = \dim \mathcal{W} < n$ then:

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{4.49}$$

Where $A_{11} \in \mathbb{R}^{k\times k}$ and $B_1 \in \mathbb{R}^{k\times m}$ and (A_{11}, B_1) controllable.

Theorem 4.12. Let $\Sigma(A, C)$ be unocontrollable, define $k = n - \dim \mathcal{W} < n$ then:

$$TAT^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad CT^{-1} = \begin{bmatrix} C-1 & 0 \end{bmatrix} \tag{4.49}$$

Where $A_{11} \in \mathbb{R}^{k\times k}$ and $C_1 \in \mathbb{R}^{p\times k}$ and (A_{11}, C_1) observable.

Change of variables: $\overline{x}(t) = Tx(t)$ s.t.

$$\Sigma(TAT^{-1}, CT^{-1}) : \begin{cases} \dot{\overline{x}}_1(t) = A_{11}\overline{x}_1(t) \\ \dot{\overline{x}}_2(t) = A_{11}\overline{x}_1(t) + A_{22}\overline{x}_2(t) \\ y(t) = C_1 x_1(t) \end{cases} \tag{4.74}$$

Controllable and observable eigenvalues:

Definition 4.9.
$\lambda \in \sigma(A)$ is called (A, B) CONTROLLABLE if

$$\text{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \tag{4.98}$$

(A, C) OBSERVABLE if

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \tag{4.99}$$

Theorem 4.15.
$\Sigma(A, B)$ in(4.1) and $\Sigma(A, C)$ in (4.25)
1) $\Sigma(A, B)$ controllable iff

$$\text{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \quad \forall \lambda \in \sigma(A) \tag{4.102}$$

2) $\Sigma(A, C)$ observable iff

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \sigma(A) \tag{4.103}$$

Theorem 4.16.
1) λ is $(A, B)-$ controllable iff λ is (TAT^{-1}, TB)- controllable.
2) λ is $(A, C)-$ observable iff λ is (TAT^{-1}, CT^{-1})- observable.

Theorem 4.17.
Let $\Sigma(A, B)$ in (4.1) be:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{4.110}$$

where $A_{11} \in \mathbb{R}^{k \times k}$ and $B_1 \in \mathbb{R}^{k \times m}$ and (A_{11}, B_1) controllable. Then $\lambda \in \sigma(A)$ is (A, B)-contrpllable iff $\lambda \notin \sigma(A_{22})$

Theorem 4.18.
Let $\Sigma(A, C)$ in (4.25) be:

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix} \tag{4.115}$$

Lecture 9:

Controllability cannonical form:

Theorem 4.13.
Let $\Sigma(A,B)$ with $m=1$ controllable then there exists a nonsingular $T\in\mathbb{R}^{n\times n}$ such that:

$$TAT^{-1}=\begin{bmatrix}0 & 1 & 0 & \dots & 0\\ & \ddots & \ddots & & \\ 0 & 0 & \dots & 0 & 1\\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1}\end{bmatrix}\quad TB=\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}\tag{4.67}$$

s.t.

$$\det(sI-A)=\triangle_A(s)=s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0=\det(sI-A)\tag{4.77}$$

Theorem 4.14.
$\Sigma(A,C)$ in (4.25) with $p=1$ observable $\Rightarrow\exists$ nonsingular T s.t.

$$TAT^{-1}=\begin{bmatrix}0 & 0 & \dots & 0 & 0 & -a_0\\ 1 & 0 & \dots & & 0 & -a_1\\ & \ddots & \ddots & & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 1 & -a_{n-1}\end{bmatrix}\qquad CT^{-1}=\begin{bmatrix}0 & 0 & \dots & 0 & 0 & 1\end{bmatrix}\tag{4.96}$$

with $a_0,\ldots,a_{n-1}\in\mathbb{R}$ and also like (4.77)

Stabilization by static state feedback:

$$\Sigma(A,B):\dot{x}(t)=Ax(t)+Bu(t)\tag{5.1}$$

In Closed loop system we have a State feedback control:

$$\begin{aligned}u(t)&=F(t)x(t)\\ \text{substitution into (5.1),gives}\dot{x}(t)&=(A+BF)x(t)\end{aligned}\tag{5.2,5.3}$$

Problem 5.1
Given $\Sigma(A,B)$ find $F\in\mathbb{R}^{m\times n}$ s.t. $\sigma(A+BF)\subset\mathbb{C}_-=\{z\in\mathbb{C}|\mathrm{Re}<0\}$
Theorem 5.1:
$A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$ given. For each monic polynomial degree n there $\exists F\in\mathbb{R}^{m\times n}$ s.t.

$$\Delta_{A+BF}(s)=p(s)\tag{5.5}$$

iff (A,B) controllable.

Corollary 5.2.
$\lambda\in\sigma(A)$ not (A,B) controllable then $\lambda\in\sigma(A+BF)$

Theorem 5.2:
Given $\Sigma(A,B)$ there exists $F\in\mathbb{R}^{m\times n}$ s.t. $\sigma(A+BF)\subset\mathbb{C}_-$ if and only if every $\lambda\in\sigma(A)$ s.t. $\lambda\notin\mathbb{C}_-$ is (A,B) controllable.

Lecture 10:

State observers:

Given

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{5.30}$$

Let $\omega(t) \in \mathbb{R}^{n_w}$ to be able to define

$$\Omega = \begin{cases} \dot{w}(t) = Pw(t) + Qu(t) + Ry(t) \\ \xi(t) = Sw(t) \end{cases} \tag{5.31}$$

Define ESTIMATION ERROR s.t.

$$e(t) = \xi(t) - x(t) \tag{5.32}$$

So when we take the derivative, and substitute $x = \xi - e = Sw - e$ we get:

$$\dot{e}(t) = (SP + SRCS - AS)\omega(t) + (A - SRC)e(t) + (SQ - B)u(t) \tag{5.34}$$

Definition 5.2.
Ω is called STATE OBSERVER: if for any $x_0 \in \mathbb{R}^n$ and $w_i \in \mathbb{R}^{n_w}$ s.t. $e(0) = Sw_0 - x_0 = 0$
For any input $u(\cdot)$ then $e(t) = 0$ for all $t \geq 0$
Definition 5.3.
STABLE STATE OBSERVER:

$$\lim_{t\to\infty} e(t) = 0 \tag{5.35}$$

Theorem 5.6:
General form of a state observer for Σ is:

$$\dot{\xi}(t) = (A - GC)\xi(t) + Bu(t) + Gy(t) \tag{5.39}$$

With $\xi(t) \in \mathbb{R}^n$ and $G = SR \in \mathbb{R}^{n\times p}$
Then the estimation error satisfies: $\dot{e}(t) = (A - GC)e(t)$ Such that the state observer $*$ is stable iff $\sigma(A - GC) \subset \mathbb{C}_-$

Detectable:

Definition 5.4.
Σ detectable if there exists $G \in \mathbb{R}^{n\times p}$ s.t. $\sigma(A - GC) \subset \mathbb{C}_-$

Lemma 5.7.
(A, C) dectectable iff (A^T, C^T) is stabilizable.

Theorem 5.8.
(5.30) detectable iff every unstable eigenvalue of A is (A, C) observable, so:

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \sigma(A) : \text{Re}(\lambda) \geq 0 \tag{5.43}$$

Corollary 5.9.
Consider (5.30),then:
1) $\exists$ stable observer
2) (A, C) detectable
3) every eigenvalue $\lambda \in \sigma(A)$ satisfying $\lambda \notin \mathbb{C}_-$ is observable.

Lecture 11:

Consider (5.30) so then we can rewrite (5.31) s.t.

$$\Gamma : \begin{cases} \dot{\omega}(t) = K\omega(t) + Ly(t) \\ u(t) = M\omega(t) + Ny(t) \end{cases} \tag{5.45}$$

Problem 5.2.
Given $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times m}$ find integer $n_w > 0$ and matrices

$$K \in \mathbb{R}^{n_w\times n_w} \quad L \in \mathbb{R}^{n_w\times p} \quad M \in \mathbb{R}^{m\times n_w} \quad N \in \mathbb{R}^{m\times p} \tag{5.48}$$

s.t. A_{cl} satisfies $\sigma(A_{cl}) \subset \mathbb{C}_{-}$. Where

$$A_{cl} = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix} \tag{5.47}$$

When we combine this with the state feedback

$$u(t) = F\xi(t) \tag{5.49}$$

so then we get

$$\Gamma : \begin{cases} \xi(t) = (A - GC + BF)\xi(t) + Gy(t) \\ u(t) = F\xi(t) \end{cases} \tag{5.50}$$

When we combine this with (5.50) s.t. $n_w = n$ we obtain

$$K = A - GC + BF \quad L = G \quad M = F \quad N = 0 \tag{5.51}$$

Lemma 5.10.
Let:
1) Ω be stable state observer for Σ
2) F solve stabilization problem for static state feedback.
Then Γ solves stabilization problem by dynamics output feedback.
Proof:
For Γ we have $A_{cl} = \begin{bmatrix} A & BF \\ GC & AGC + BF \end{bmatrix}$ define $T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ so then $TA_{cl}T^{-1} = \begin{bmatrix} A + BF & BF \\ 0 & A - GC \end{bmatrix}$
Then $\sigma(A_{cl}) = \sigma(A + BF) \cup \sigma(A - GC)$
But by 1; $\sigma(A - GC) \subset \mathbb{C}_{-}$ and by 2; $\sigma(A + BF) \subset \mathbb{C}_{-}$

Theorem 5.11:
Consider (5.30). There exists Γ of the form (5.45) iff (A, B) stabilizable and (A, C) detectable.
Detectability:
Let,$\lambda \in \sigma(A)$ be not (A, C) observable, so $\exists p \in \mathbb{C}^n$ where $p \neq 0$ s.t. $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} p = 0$ Consider two trajectories of controlled Σ:
1) $x_0 = 0$ where w_0 arbitrary then $x(\cdot)$ satisfies: $\lim_{t \to \infty} x(t) = 0$
2) $x_0 = p$ where w_0 arbitrary.
Claim: $x'(t) = x(t) + e^{\lambda t} p$
Namely:$\dot{x}'(t) = \dot{x}(t) + te^{\lambda t} p = Ax(t) + Bu(t) + Ae^{\lambda t} p = A(x(t) + e^{\lambda t} p) + Bu(t) = Ax](t) + Bu(t)$
Also $Cx'(t) = Cx(t) + Ce^{\lambda t} p$ so $Cx'(t) = Cx(t)$
So $u(t) = u'(t)$

However $\lim_{t \to \infty} x(t) = 0$ so $\lim_{t \to \infty} x(t) + e^{\lambda t} p = 0$ so $e^{\lambda t} p \xrightarrow{t \to \infty} 0 \Rightarrow \text{Re}(\lambda) < 0 \Leftrightarrow \lambda \in \mathbb{C}_-$

Lecture 12:

Define

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{6.1}$$

Definition 6.1.
Two systems $\Sigma(A, B, C, D)$ and $\Sigma(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ and SIMILAR if

$$\overline{A} = TAT^{-1} \quad \overline{B} = TB \quad \overline{C} = CT^{-1} \quad \overline{D} = D \tag{6.5}$$

Theorem 6.1:
$\Sigma(A, B, C, D)$ and $\Sigma(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ be similar, then:
$y(t; 0, u) = \overline{y}(t; 0, u)$ for any $u : I \to \mathbb{R}^m$.
Lemma 6.2:
Consider

$$y_\varepsilon(t) = \int\limits_{-\varepsilon}^{t} Ce^{A(t-\tau)} Bu_\varepsilon(\tau)d\tau \tag{6.10}$$

Then

$$\lim_{\varepsilon \to 0^+} y_\varepsilon(t) = \begin{cases} Ce^{At}B \quad t > 0 \\ 0\, t < 0 \end{cases} \tag{5}$$

PROOF:
$t > 0$ for $t \geq \varepsilon$ so then:
$y_\varepsilon(t) = \int\limits_{-\varepsilon}^{t} Ce^{A(t-\tau)}Bu_\varepsilon(\tau)d\tau = \frac{1}{2\varepsilon} \int\limits_{-\varepsilon}^{\varepsilon} Ce^{A(T-\tau)}Bd\tau = \frac{1}{2\varepsilon} \int\limits_{-\varepsilon}^{\varepsilon} Ce^{At}e^{-A\tau}Bd\tau = \frac{1}{2\varepsilon}Ce^{At}[I\tau - \frac{A\tau^2}{2!} + \frac{A^2t^3}{3!} + \ldots +]_{-\varepsilon}^{\varepsilon}Bd\tau = \frac{1}{2\varepsilon}Ce^{At}(I(2\varepsilon) + 2\frac{A^2\varepsilon^3}{3!} + \ldots)B$
$\lim\limits_{\varepsilon \to 0+} y_\varepsilon(t) = Ce^{At}B$
Remark: Function ,$\delta : \mathbb{R} \to \mathbb{R}$ with defining properties:
1 $\delta(t) = 0$ for all $t \neq 0$
2 For any continuous function

$$\phi \Rightarrow \int\limits_{-\infty}^{\infty} \phi(t - \tau)\delta(\tau) = \phi(t) \tag{6.17}$$

More explicity:

$$\int\limits_{-\infty}^{\infty} \phi(t - \tau)\delta(\tau)d\tau = \lim_{\varepsilon \to 0} \int\limits_{-\infty}^{\infty} \phi(t - \tau)u_\varepsilon(\tau)d\tau \tag{6.19}$$

Definition 6.2:
For (6.1) IMPULSE RESPOSE MATRIX defined as:

$$H(t) = \begin{cases} Ce^{At}B + D\delta(t), \quad t \geq 0 \\ 0 \quad t < 0 \end{cases} \tag{6.22}$$

Theorem 6.3: Consider system with above impulse response matrix, for $t \geq 0$ output given by

$$y(t; 0, u) = \int\limits_{0}^{t} H(t - \tau)u(\tau)d\tau \tag{6.23}$$

Theorem 6.4.
When two systems are similar, then for their respectively impulse response matrices, we have that $H(t) = \overline{H}(t)$ for all $t \in \mathbb{R}$

Theorem 6.5
Consider (6.1) and (6.22) then:
Option 1:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$
$$\Rightarrow H(t) = \begin{cases} C_1 e^{A_{11}t} B_1 + D\delta(t) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{6.30,6.31}$$

Option 2:

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$
$$\Rightarrow H(t) = \begin{cases} C_1 e^{A_{11}t} B_1 + D\delta(t) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{6.32,6.33}$$

Remark 6.3.
We can define output function as

$$y(t) = \int_{-\infty}^{\infty} H(t, \tau) u(\tau) d\tau \tag{6.35}$$

Furthermore, this system is:
1) Casual (or non anticipating) if $H(t, \tau) = 0$ for all $\tau > t$. So t only depends on past input values.
2) time-invariant if $H(t + s, \tau + s) = H(t, \tau)$

Definition 6.3.
$f : \mathbb{R}_+ \to \mathbb{R}$ exponentially bounded (so $|f(t)| \leq Me^{\alpha t}$) with bounding exponent α then Laplace transform:

$$\mathcal{L}(f)(s) = \int_0^{\infty} f(t) e^{-st} dt \tag{6.58}$$

Theorem 6.6.
1)

$$\mathcal{L}(f + f') = \mathcal{L}(f) + \mathcal{L}(f') \qquad \mathcal{L}(\alpha f) = \alpha \mathcal{L}(f) \tag{6.62}$$

for any bounded $f, f' : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in \mathbb{C}$
2) $f : \mathbb{R}_+ \to \mathbb{R}$ continuous differential, $\dot{f}$ exponentially bounded then f exponentially bounded

$$\mathcal{L}(\dot{f}) = s\mathcal{L}(f) - f(0) \tag{6.63}$$

3) $u, h : \mathbb{R}_+ \to \mathbb{R}$ exponentially bounded, then

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau \tag{6.64}$$

is exponentially bounded

$$\mathcal{L}(y) = \mathcal{L}(h)\mathcal{L}(u) \tag{6.65}$$

Definition 6.4.
TRANSFER FUNCTION MATRIX consider (6.1) then

$$T(s) = C(sI - A)^{-1}B + D \tag{6.71}$$

Theorem 6.7.
We see that

$$T(s) = \mathcal{L}(H)(s) \tag{6.72}$$

for all s s.t. $\mathrm{Re}(s) > \Lambda(A)$ with $\Lambda(A)$ the spectral abscis [2] of A
$\Lambda(A) = \max\{\mathrm{Re}(\lambda) | \lambda \in \sigma(A)\}$

Theorem 6.8.
When two systems are similar then

$$T(s) = \overline{T}(s) \quad \forall s \in \mathbb{C} \tag{6}$$

Lecture 13:

Cramer's rule:
$(sI - A)^{-1} = \frac{1}{\Delta_A(s)}\text{adj}(sI - A)$

Single-input single-output system:

$$\Sigma_{\text{SISO}} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{6.97}$$

Transfer function as rational function:

$$T(s) = \frac{p_m s^m + \ldots + p_1 s + p_0}{q_s^n + \ldots + q_1 + q_0} \tag{6.98}$$

Siso: single input, single output.
$\Sigma_{\text{Siso}} \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$

Theorem 6.9.
Consider Σ_{siso} with

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots & \\ 0 & & & & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} c_0 & 1 & \ldots & c_{n-2} & c_{n-1} \end{bmatrix} \tag{6.99}$$

then:

$$T(s) = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \tag{6.100}$$

Proof:
Compute

$$\text{adj}(sI - A) = \begin{bmatrix} * & \ldots & * & 1 \\ * & \ldots & * & s \\ \vdots & & \vdots & \vdots \\ * & \ldots & * & s^{n-1} \end{bmatrix}$$
$$\Rightarrow C\text{Adj}(sI - A)B = c_{n-1}s^{n-1} + \ldots + c_1 s + c_0 \qquad (6.101,6.103)$$

Combining this with Cramer's rule and $\Delta_A(s)$ we get indeed the result we wanted.

Theorem 6.10.
Consider (6.97) and let (A, B) be controllable, then:

$$p(s) = C\text{adj}(sI - A)B \quad q(s) = \Delta_A(s) \tag{6.116}$$

And coprime iff (A, C) observable.

PROOF:
(A, c) nonobservable $\Rightarrow \exists \mathbf{v} \in \mathbb{C}^n$ where $\mathbf{v} \neq 0$ s.t., $\left[\begin{smallmatrix} A-\lambda I \\ C \end{smallmatrix}\right] \mathbf{v} = 0$ for some $\lambda \in \sigma(A) \Rightarrow \mathbf{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T$ then:

$$Av = \lambda v \Rightarrow \begin{cases} v_2 = \lambda v_1 \\ v_3 = \lambda v_2 \\ \vdots \\ v_n = \lambda v_{n-1} \end{cases} \Rightarrow -a_0 v_1 - a_1 v_2 - \dots - a_{n-1} v_n = \lambda v_n \Rightarrow \star$$

Let $v = v_1 \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$ for $v_1 \neq 0 \xrightarrow{\text{substitute in } \star} 0 = (\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0)v_1 = \Delta_A(\lambda)v_1$
So λ is a root of $\Delta_A = q$
Using that $0 = Cv = (c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \dots + c_1\lambda + c_0)v_1 = c\text{adj}(\lambda I - A)Bv_1 = p(\lambda)v_1$
So λ also root of p
So p and q not coprime, what concludes the proof.

Theorem 6.11:
Consider (6.97) with

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -a_0 \\ 1 & 0 & & & 0 & -a_1 \\ 0 & 1 & & & 0 & -a_2 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & & & \ddots & 1\ 0 & -a_{n-2} \\ 0 & 0 & \dots & 0 & 1 & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} \quad C^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{6.129}$$

Then we have that

$$T(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} \tag{6.130}$$

Note that (A, C) in (6.129) observable so therefore the matrices in (6.129) observability canonical form.

Theorem 6.12.
Consider (6.97) and let (A, C) observable, then

$$p(s) = C\text{adj}(sI - A)B \quad q(s) = \Delta_A(s) \tag{6.131}$$

are coprime iff (A, B) controllable.

Lecture 14:

For SISO:

$$T(s) = \frac{p(s)}{q(s)} = \frac{p'(s)}{q'(s)} \tag{6.132}$$

where q' and p' are coprime.
Non-SISO:

$$T(s) = \frac{1}{q(s)} P(s) \tag{6.133}$$

where P matrix of polynomials.
Definition 6.5.
POLE: $\lambda \in \mathbb{C}$ pole of $T(s) = \frac{p(s)}{q(s)}$ if it is a root of $q'(s)$
$\lambda \in \mathbb{C}$ is a pole of $T(s) = \frac{1}{q(s)} P(s)$ if it is a pole of at least one of its elements.

Theorem 6.13.
Consider (6.1) and its transfer matrix T then:
1) $\lambda \in \mathbb{C}$ pole of T then $\lambda \in \sigma(A)$
2) $\lambda \in \sigma(A)$ and (A, B) controllable and (A, C) observable, then λ pole of T

Definition 6.6.
$\Sigma(A, B, C, D)$ is EXTERNALLY STABLE if $\exists \gamma > 0$ s.t. for any bounded $u : \mathbb{R}_+ \to \mathbb{R}^m$

$$\sup_{t \in \mathbb{R}_+} \|y(t; 0, u)\|_2 \leq \gamma \sup_{t \in \mathbb{R}_+} \|u(t)\|_2 \tag{6.134}$$

EQUIVALENT DEFINITION:

$$\|u(t)\|_2 \leq 1 \forall t \in \mathbb{R}_+ \Rightarrow \|y(t[0, u)\|_2 \leq \gamma \forall t \in \mathbb{R}_+ \tag{6.135}$$

Lemma 6.14.
$\Sigma(A, B, C, D)$ in (6.1) externally stable iff $\Sigma(A, B, C, D)$ is externally stable.

Theorem 6.15:
For (6.1) and its transfer T the following are equivalent:
1) Σ externally stable.
2) $\int_0^\infty \|Ce^{At}B\| dt \leq \infty$
3) $\lim_{t \to \infty} Ce^{At}B = 0$
4) all poles of T are in $\mathbb{C}_-$
PROOF:
In Lecture notes

Definition 6.7.
(6.1) is internally stable if (3.1) is asymptotically stable.

Theorem 6.16:
Consider (6.1) then:
1) Internally stable, then externally stable.
2) Externally stable,(A, B) controllable, (A, C) observable $\Rightarrow$ internally stable.